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An extension of Jellett's theorem

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Abstract

The aim of this work is to show that a star-shaped hypersurface of constant mean curvature into the Euclidean sphere $\mathbb{S}^{n+1}$ must be a geodesic sphere. This result extends the one obtained by Jellett in 1853 for such type of surfaces in the Euclidean space $\mathbb{R}^3$. In order to do that we will compute a useful formula for the Laplacian of a new support function defined over a hypersurface M of a Riemannian manifold $\overline{M}$.

MSC: primary 53C42, 53C45; secondary 53C65

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1. Introduction

On the middle of the eighteenth century Jellett [9] showed that a star-shaped constant mean curvature surface $\Sigma \subset \mathbb{R}^3$ is a round sphere. It is important to point out that Jellett already used the so-called Minkowisky formula to obtain his result. Later, Hopf [6] proved a generalization of this theorem showing that an immersed constant mean curvature surface $\Sigma \subset \mathbb{R}^3$ homeomorphic to a sphere is also a round sphere. On the other hand, for hypersurfaces $\Sigma^n \subset \mathbb{R}^{n+1}$ we may consider the r-th symmetric function of the principal curvatures, which is denoted by H_r. In 1952, Süss [15] proved that compact convex hypersurfaces in the Euclidean space with some H_r constant must be round spheres. The convexity condition was improved by Hsiung [8], who showed that a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ whose classical support function has a well defined sign and having any symmetric function of the principal curvatures constant must be a round sphere. Later, in 1956 Alexandrov [2] proved that a compact embedded constant mean curvature

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hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is also a round sphere. On the eighties of the last century Hsiang, Teng and Yu [7] built examples of spherical immersed hypersurfaces of constant mean curvature in $\mathbb{R}^{2n}$ that are not round spheres. After that, Ros extended Alexandrov result in [12] and [13], showing that a round sphere is the unique compact embedded hypersurface in the Euclidean space, provided any symmetric function H_r is constant. In the sequel Montiel and Ros [11] extended this result for any compact embedded hypersurface in the hyperbolic space $\mathbb{H}^{n+1}$ as well as in an open hemisphere of the Euclidean sphere $\mathbb{S}^{n+1}$. On the other hand it is well known that products of spheres produce hypersurfaces in the Euclidean sphere with H_r constant for any $r = 1, \dots, n$. Therefore for hypersurfaces contained in the Euclidean sphere $\mathbb{S}^{n+1}$ we have a lot of examples with H_r constant which are not round spheres. However, if we go back until Jellett's idea we may obtain a similar result in the Euclidean sphere for star-shaped hypersurfaces of constant mean curvature without assuming it is contained in an open hemisphere. More precisely, we will prove the following theorem.

Theorem 1. *Let $\Sigma^n \subset \mathbb{S}^{n+1}$ be a star-shaped hypersurface of constant mean curvature. Then Σ^n is a geodesic sphere.*

2. A support function

In order to show Theorem 1 we will compute the Laplacian of a support function which was obtained by Sousa on [14]. This result is a generalization of a theorem contained in a paper due to Fornari and Ripoll [5]. First we recall that for a r-covariant tensor field ω and for a vector field $V \in \chi(N)$ the Lie derivative of ω with respect to V, denoted by $L_V\omega$, is defined according to the law:

$$(L_V\omega)(X_1, \dots, X_r) = V\big(\omega(X_1, \dots, X_r)\big) - \sum_{i=1}^{r} \omega\big(X_1, \dots, [V, X_i], \dots, X_r\big).$$

Now we consider $\overline{M}^{n+1}$ a Riemannian manifold and V a vector field of $\overline{M}^{n+1}$. Let M^n be a hypersurface of $\overline{M}^{n+1}$ and N a unit vector field normal to M^n. Define the support function $f(p) = \langle N(p), V(p)\rangle$, $p \in M^n$. Let A be the second fundamental form of M^n. Given $p \in M^n$, let $\{e_1(p), \dots, e_n(p)\} \subset T_pM$ be an orthonormal basis diagonalizing A at p, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues associated to $e_1(p), \dots, e_n(p)$, respectively. Denoting the Ricci curvature of $\overline{M}^{n+1}$ by Ric we introduce the Jacobi operator $\mathcal{L}$ which is given by

$$\mathcal{L} = \Delta + |A|^2 + Ric(N),$$

where Δ stands for the Laplacian of M^n in the induced metric, which will be denoted by g.

We put $L_{N,N} = (L_V g)(N, N)$, $L_{i,i} = (L_V g)(e_i, e_i)$ and $L_{i,N} = (L_V g)(e_i, N)$. With this notation we have the following theorem.

Theorem 2. *Let $\overline{M}^{n+1}$ be a Riemannian manifold and let V be a vector field of $\overline{M}^{n+1}$. Let M^n be a hypersurface of $\overline{M}^{n+1}$ and let N be a unit vector field normal to M^n in $\overline{M}^{n+1}$. Under the above notations we have*

$$\mathcal{L}f = -n\langle V, \nabla H\rangle - \frac{n}{2} H L_{N,N} - \frac{1}{2}\sum_{i=1}^{n} N(L_{i,i}) + \sum_{i=1}^{n} e_i(L_{i,N}),$$

where ∇H stands for the gradient of the mean curvature of M.

Proof. Given $p \in M^n$, let $\{e_1(p), \dots, e_n(p)\} \subset T_pM^n$ be an orthonormal basis diagonalizing A at p, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues associated to $e_1(p), \dots, e_n(p)$, respectively. Denote by $e_1, \dots, e_n$ the geodesic frame that extends $e_1(p), \dots, e_n(p)$ to a neighborhood of p in M^n. We may extend $e_{i's}$ and N to a neighborhood of p in $\overline{M}^{n+1}$ in such way that $\overline{\nabla}_N e_i(p) = 0$. Then we have

$$\Delta f(p) = \sum_{i=1}^n e_i e_i(f)(p).$$

Now we notice that

$$e_i e_i(f) = \langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N, V\rangle + 2\langle \overline{\nabla}_{e_i} N, \overline{\nabla}_{e_i} V\rangle + \langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V, N\rangle.$$

Since $\overline{\nabla}_{e_i} N(p) = -A(e_i(p)) = -\lambda_i e_i(p)$ we obtain

$$\langle \overline{\nabla}_{e_i} N, \overline{\nabla}_{e_i} V\rangle(p) = -\lambda_i \langle e_i, \overline{\nabla}_{e_i} V\rangle(p) = -\frac{\lambda_i}{2} L_{i,i}(p).$$

Then

$$e_i e_i(f)(p) = \langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N, V\rangle(p) + \langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V, N\rangle(p) - \lambda_i L_{i,i}(p). \tag{2.1}$$

Computing $e_i(L_{i,N})$ we obtain

$$e_i(L_{i,N}) = \langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V, N\rangle + \langle \overline{\nabla}_{e_i} V, \overline{\nabla}_{e_i} N\rangle + \langle \overline{\nabla}_{e_i} \overline{\nabla}_N V, e_i\rangle + \langle \overline{\nabla}_N V, \overline{\nabla}_{e_i} e_i\rangle.$$

Taking into account that $(\overline{\nabla}_{e_i} e_i(p))^\top = 0$ we have $\overline{\nabla}_{e_i} e_i(p) = \lambda_i N(p)$. Hence we get

$$\langle \overline{\nabla}_{e_i} V, \overline{\nabla}_{e_i} N\rangle(p) = -\lambda_i \langle \overline{\nabla}_{e_i} V, e_i\rangle(p) = -\frac{\lambda_i}{2} L_{i,i}(p)$$

and

$$\langle \overline{\nabla}_N V, \overline{\nabla}_{e_i} e_i\rangle(p) = \lambda_i \langle \overline{\nabla}_N V, N\rangle(p) = \frac{\lambda_i}{2} L_{N,N}(p).$$

Thus we obtain, at the point p,

$$\langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V, N\rangle = -\langle \overline{\nabla}_{e_i} \overline{\nabla}_N V, e_i\rangle + e_i(L_{i,N}) + \frac{\lambda_i}{2} L_{i,i} - \frac{\lambda_i}{2} L_{N,N}. \tag{2.2}$$

Next we may use the curvature tensor of $\overline{M}$. To do that we note that $\overline{\nabla}_N e_i(p) = 0$ yields $[N, e_i] = -\overline{\nabla}_{e_i} N = \lambda_i e_i$ at p. Then

$$\langle \overline{\nabla}_{[N,e_i]} V, e_i\rangle(p) = \lambda_i \langle \overline{\nabla}_{e_i} V, e_i\rangle(p) = \frac{\lambda_i}{2} L_{i,i}(p). \tag{2.3}$$

Since $\frac{1}{2} L_{i,i} = \langle \overline{\nabla}_{e_i} V, e_i\rangle$ and $\overline{\nabla}_N e_i(p) = 0$ we have

$$\frac{1}{2} N(L_{i,i})(p) = \langle \overline{\nabla}_N \overline{\nabla}_{e_i} V, e_i\rangle(p). \tag{2.4}$$

It follows from Eqs. (2.3) and (2.4) that, at p,

$$\langle \overline{R}(N, e_i)V, e_i\rangle = \langle \overline{\nabla}_{e_i} \overline{\nabla}_N V, e_i\rangle - \frac{1}{2} N(L_{i,i}) + \frac{\lambda_i}{2} L_{i,i}. \tag{2.5}$$

Therefore, making use of Eqs. (2.1), (2.2) and (2.5) we have also, at p,

$$e_i e_i(f) = -\langle \overline{R}(N, e_i)V, e_i\rangle + \langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} N, V\rangle + e_i(L_{i,N}) - \frac{1}{2} N(L_{i,i}) - \frac{\lambda_i}{2} L_{N,N}. \tag{2.6}$$

Writing $V=\sum_{j=1}^{n} v_j e_j + fN$ we have

$$\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N, V\rangle = \sum_{j=1}^{n} v_j \langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N, e_j\rangle + f\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N, N\rangle. \tag{2.7}$$

Differentiating now $\langle N, N\rangle$ and $\langle N, e_j\rangle$ twice with respect to e_i, we obtain respectively

$$\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N, N\rangle(p) = -\langle \overline{\nabla}_{e_i}N, \overline{\nabla}_{e_i}N\rangle(p) = -\lambda_i^2$$

and

$$\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N, e_j\rangle + 2\langle \overline{\nabla}_{e_i}N, \overline{\nabla}_{e_i}e_j\rangle + \langle N, \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}e_j\rangle = 0.$$

Since $\overline{\nabla}_{e_i}N = -\lambda_i e_i$ we have $\langle \overline{\nabla}_{e_i}N, \overline{\nabla}_{e_i}e_j\rangle = -\lambda_i\langle e_i, \nabla_{e_i}e_j\rangle$, so that

$$\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N, e_j\rangle(p) = -\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}e_j, N\rangle(p). \tag{2.8}$$

Taking into account that $\langle \overline{\nabla}_{e_i}e_j, N\rangle = \langle \overline{\nabla}_{e_j}e_i, N\rangle$ we compute the derivative of both members above with respect to e_i to arrive at

$$\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}e_j, N\rangle + \langle \overline{\nabla}_{e_i}e_j, \overline{\nabla}_{e_i}N\rangle = \langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_j}e_i, N\rangle + \langle \overline{\nabla}_{e_j}e_i, \overline{\nabla}_{e_i}N\rangle.$$

We use now that $(\overline{\nabla}_{e_i}e_j(p))^{\top} = (\overline{\nabla}_{e_j}e_i(p))^{\top} = 0$ to get

$$\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}e_j, N\rangle(p) = \langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_j}e_i, N\rangle(p). \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N, e_j\rangle(p) = -\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_j}e_i, N\rangle(p). \tag{2.10}$$

On the other hand $[e_i, e_j](p) = (\overline{\nabla}_{e_i}e_j(p) - \overline{\nabla}_{e_j}e_i(p))^{\top} = 0$ and $\overline{\nabla}_{e_i}e_i(p) = (\lambda_i N)(p)$ yield

$$e_j\langle \overline{\nabla}_{e_i}e_i, N\rangle(p) = \langle \overline{\nabla}_{e_j}\overline{\nabla}_{e_i}e_i, N\rangle(p). \tag{2.11}$$

Since $\langle \overline{R}(e_i, e_j)e_i, N\rangle = \langle \overline{\nabla}_{e_j}\overline{\nabla}_{e_i}e_i - \overline{\nabla}_{e_i}\overline{\nabla}_{e_j}e_i + \overline{\nabla}_{[e_i,e_j]}e_i, N\rangle$ we have from (2.10) and (2.11) that

$$\begin{aligned}\langle \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N, e_j\rangle(p) &= \big\langle \overline{R}(e_i, e_j)e_i, N\big\rangle(p) - \langle \overline{\nabla}_{e_j}\overline{\nabla}_{e_i}e_i, N\rangle(p)\\ &= \big\langle \overline{R}(e_i, e_j)e_i, N\big\rangle(p) - e_j\langle \overline{\nabla}_{e_i}e_i, N\rangle(p).\end{aligned}$$

Then from (2.7) we have, at p,

$$\begin{aligned}\langle V, \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}N\rangle &= \sum_{j=1}^{n} v_j\big(\big\langle \overline{R}(e_i, e_j)e_i, N\big\rangle - e_j\langle \nabla_{e_i}e_i, N\rangle\big) - f\lambda_i^2\\ &= \sum_{j=1}^{n} v_j\big\langle \overline{R}(e_i, e_j)e_i, N\big\rangle - \sum_{j=1}^{n} v_j e_j\langle \overline{\nabla}_{e_i}e_i, N\rangle - f\lambda_i^2\\ &= \left\langle \overline{R}\left(e_i, \sum_{j=1}^{n} v_j e_j\right)e_i, N\right\rangle - \sum_{j=1}^{n} v_j e_j\langle \overline{\nabla}_{e_i}e_i, N\rangle - f\lambda_i^2\\ &= \big\langle \overline{R}(e_i, V - fN)e_i, N\big\rangle - \sum_{j=1}^{n} v_j e_j\langle \overline{\nabla}_{e_i}e_i, N\rangle - f\lambda_i^2\\ &= \big\langle \overline{R}(e_i, V)e_i, N\big\rangle - f\big\langle \overline{R}(e_i, N)e_i, N\big\rangle - f\lambda_i^2 - \sum_{j=1}^{n} v_j\big(e_j\langle \overline{\nabla}_{e_i}e_i, N\rangle\big).\end{aligned}$$

Therefore, at p, we have

$$e_i e_i(f) = e_i(L_{i,N}) - \frac{1}{2}N(L_{i,i}) - \frac{\lambda_i}{2}L_{N,N} - f\lambda_i^2 \\ - f\langle \overline{R}(e_i, N)e_i, N\rangle - \sum_{j=1}^{n}\langle V, e_j\rangle\big(e_j\langle\overline{\nabla}_{e_i}e_i, N\rangle\big).$$

Since $nH = \sum_{i=1}^{n}\langle\overline{\nabla}_{e_i}e_i - (\overline{\nabla}_{e_i}e_i)^{\top}, N\rangle = \sum_{i=1}^{n}\langle\overline{\nabla}_{e_i}e_i, N\rangle$, we obtain

$$\mathcal{L}f = -n\langle V, \nabla H\rangle - \frac{n}{2}HL_{N,N} - \frac{1}{2}\sum_{i=1}^{n}N(L_{i,i}) + \sum_{i=1}^{n}e_i(L_{i,N}),$$

which finishes the proof of the theorem. □

3. Conformal Killing vector field

Now we remember that $V \in \chi(\overline{M})$ is a conformal Killing vector field if there exists a differentiable function ψ on $\overline{M}$ such that

$$L_V g = 2\psi g,$$

where g stands for the metric of $\overline{M}$.

Therefore if $V \in \chi(\overline{M})$ is a conformal Killing vector field we obtain $(L_V g)(N, N) = 2\psi$, $(L_V g)(e_i, e_i) = 2\psi$ and $(L_V g)(e_i, N) = 2\psi g(e_i, N) = 0$. Since $L_{i,i} = 2\psi$ we also have

$$\frac{1}{2}\sum_{i=1}^{n}N(L_{i,i}) = nN(\psi).$$

As a consequence of Theorem 2 we derive the following theorem which was obtained by Fornari and Ripoll [5] for the particular case when $\psi = 0$. In the presented version of it also appears on Alias, Dajczer and Ripoll [3] in the Riemannian case, whereas for the Lorentzian case it appears on Barros, Brasil and Caminha [4].

Theorem 3. *Let M^n be a hypersurface of a Riemannian manifold $\overline{M}^{n+1}$ and let V be a conformal vector field on $\overline{M}^{n+1}$. If N is a unit vector field normal to M^n in $\overline{M}^{n+1}$ and $f(p) = \langle N, V\rangle(p)$, $p \in M^n$, then*

$$\mathcal{L}f = -n\langle V, \nabla H\rangle - n\big(\psi H + N(\psi)\big),$$

where H stands for the mean curvature of M^n and ∇H is the gradient of H.

Next we consider $\overline{M}_c^{n+1}$ a space form, $p_o \in \overline{M}_c^{n+1}$ and $d : \overline{M}_c^{n+1} \to \mathbb{R}$ the distance function relative to p_o. According to Alencar and Frensel [1] the position vector on $\overline{M}_c^{n+1}$ relative to the basis point p_o is given by $V = s(d)\nabla d$, where $s(t)$ is solution of the equation $y'' + cy = 0$, under the initial conditions $y(0) = 0$ and $y'(0) = 1$.

One of the notable properties of V is that it is a conformal Killing vector field with conformal factor $\psi = s'(d)$. In fact,

$$\langle\overline{\nabla}_X V, Y\rangle = s'(d)\langle\nabla d, X\rangle\langle\nabla d, Y\rangle + s(d)\langle\overline{\nabla}_X\nabla d, Y\rangle.$$

According to a result due to Jorge and Koutroufiotis [10] we have

$$\langle \overline{\nabla}_X \nabla d, Y\rangle = \frac{s'(d)}{s(d)}\big(\langle X, Y\rangle - \langle \nabla d, X\rangle\langle \nabla d, Y\rangle\big), \tag{3.1}$$

for any vector fields $X, Y \in \chi(\overline{M}^{n+1}_c)$.

Therefore we arrive at

$$\langle \overline{\nabla}_X V, Y\rangle = s'(d)\langle X, Y\rangle,$$

which implies

$$L_V g(X, Y) = 2s'(d)\langle X, Y\rangle.$$

From where we finish our claim.

Next we present an immediate consequence of Theorem 3.

Corollary 1. *Let $\overline{M}^{n+1}_c$ be a space form and V the position vector field on $\overline{M}^{n+1}_c$ relative to a fixed point p_0. Let M^n be a constant mean curvature hypersurface of $\overline{M}^{n+1}_c$ and let N be a unit normal vector field to M^n in $\overline{M}^{n+1}_c$. If $f = \langle N, V\rangle$ then*

$$\Delta f = -|A|^2 f - nHs'(d),$$

where H and $|A|$ are respectively the mean curvature and the norm of the second fundamental form of M.

Proof. First of all we notice that using the fact H is constant and $\psi = s'(d)$ we have from Theorem 3

$$\mathcal{L}f = -nHs'(d) - nN\big(s'(d)\big).$$

On the other hand $s''(d) = -cs(d)$ yields $-nN(s'(d)) = cn\langle s(d)\nabla d, N\rangle = cnf$. But, taking into account that $Ric(N) = cn$ and $\mathcal{L}f = \Delta f + |A|^2 f + Ric(N)f$ we obtain

$$\Delta f + |A|^2 f + Ric(N)f = -nHs'(d) + Ric(N)f.$$

From where we conclude the proof of the corollary. □

On the other hand Alencar and Frensel [1] have proved the following proposition.

Proposition 1. *Let $x : M^n \to \overline{M}^{n+1}_c$ be an isometric immersion of a compact Riemannian manifold M^n into a space form $\overline{M}^{n+1}_c$. Then*

$$\int_M Hf\,dM = -\int_M s'(d)\,dM,$$

where H is the mean curvature of M^n.

4. Radial graphs

Let $M^n \subset \mathbb{R}^{n+1}$ be a radial graph defined over an Euclidean sphere $\mathbb{S}^n(r)$ of radius $r > 0$ that corresponds to a star-shaped hypersurface. We introduce coordinates $u = (u_1, \dots, u_n)$ and let $X(u)$ and $Y(u)$ be parametrizations of $\mathbb{S}^n(r)$ and M^n, respectively. If $\rho(u) = |Y(u)| > 0$, then $Y = \rho X$.

Now we take $f : M^n \to \mathbb{R}$ defined by $f(Y) = \langle Y, N_Y \rangle$, where N_Y is a unit vector field normal to M^n. Letting $\frac{\partial h}{\partial u_i} = h_i$ we have $Y_i = \rho X_i + \rho_i X$. From where we obtain

$$\left\langle \rho X, \frac{Y_1 \wedge \cdots \wedge Y_n}{|Y_1 \wedge \cdots \wedge Y_n|} \right\rangle = \left\langle \rho X, \frac{(\rho X_1) \wedge \cdots \wedge (\rho X_n)}{|Y_1 \wedge \cdots \wedge Y_n|} \right\rangle,$$

so that

$$\begin{aligned} f(Y) &= \left\langle \rho X, \frac{Y_1 \wedge \cdots \wedge Y_n}{|Y_1 \wedge \cdots \wedge Y_n|} \right\rangle = \rho^{n+1} \frac{|X_1 \wedge \cdots \wedge X_n|}{|Y_1 \wedge \cdots \wedge Y_n|} \langle X, N_X \rangle \\ &= -\frac{\rho^{n+1}}{r} \frac{|X_1 \wedge \cdots \wedge X_n|}{|Y_1 \wedge \cdots \wedge Y_n|} \langle X, X \rangle = -r\rho^{n+1} \frac{|X_1 \wedge \cdots \wedge X_n|}{|Y_1 \wedge \cdots \wedge Y_n|} < 0. \end{aligned}$$

Here we are considering $N_X = -\frac{1}{r} X$ as the unit normal vector field to $\mathbb{S}^n(r)$ in such way that its mean curvature is strictly positive.

A similar construction is done for radial graph $M^n \subset \mathbb{S}^{n+1}$. In fact, we fixed a point $p_0 \in \mathbb{S}^{n+1}$, which corresponds to the origin, and for each direction $v \in T_{p_0}\mathbb{S}^{n+1}$ we consider a point $p(v) \in M^n$ that corresponds to the end point of the geodesic segment on $\mathbb{S}^{n+1}$ starting from p_0 in the direction of v.

Given a complete radial graph $M^n \subset \mathbb{S}^{n+1}$ as above we consider the stereographic projection $\pi : \mathbb{S}^{n+1} \backslash \{p_0\} \to \mathbb{R}^{n+1}$ and let $V_{\mathbb{S}^{n+1}}$ be the position vector field with basis point p_0 on $\mathbb{S}^{n+1}$. Hence we have the next lemma.

Lemma 1. *Under the above conditions the function $f = \langle V_{\mathbb{S}^{n+1}}, N_Y \rangle$ has a well defined sign.*

Proof. Let X and Y be parametrizations of $\mathbb{S}^n(r)$ and M^n respectively. Then $\pi(X)$ is a parametrization of a sphere on $\mathbb{R}^{n+1}$ while $\pi(Y)$ is a parametrization of a radial graph over $\pi(X)$ on $\mathbb{R}^{n+1}$.

Let $g_1, g_2 : \pi(Y) \to \mathbb{R}$ be continuous function such that either $g_i > 0$ or $g_i < 0$, $d\pi(V_{\mathbb{S}^{n+1}}) = g_1 V_{\mathbb{R}^{n+1}}$ and $d\pi(N_Y) = g_2 N_{\pi(Y)}$. Let e^{ϕ} be the conformal factor of π. Then

$$e^{2\phi} \langle V_{\mathbb{S}^{n+1}}, N_Y \rangle = \langle d\pi(V_{\mathbb{S}^{n+1}}), d\pi(N_Y) \rangle = \langle g_1 V_{\mathbb{R}^{n+1}}, g_2 N_{\pi(Y)} \rangle > 0 \quad (\text{or} < 0).$$

From where we derive the desired result. $\square$

5. Proof of Theorem 1

According to Corollary 1 of Theorem 3 we have

$$\int_M |A|^2 f \, dM = -\int_M nHs'(d) \, dM.$$

Taking into account that H is constant, it follows from Proposition 1 that

$$-\int_M nHs'(d) \, dM = \int_M nH^2 f \, dM.$$

Comparing the last two equations we arrive at

$$\int_M |A|^2 f \, dM = \int_M nH^2 f \, dM.$$

Now we remember that $|A|^2 \geqslant nH^2$, occurring equality if and only if M^n is totally umbilical. Since f has a well defined sign we complete the proof of the result.

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